Convexity of Some Spectral Functions on Hermitian Matrices.

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Mathematical Analysis

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Abstract

We prove in this note the convexity of the functions $u \circ \lambda$ and more generally $u \circ \lambda_B$ on the space of Hermitian matrices, for B a fixed positive definite hermitian matrix, when $u : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is a symmetric convex function which is lower semi-continuous on $\mathbb{R}^m$, and finite in at least one point of $\mathbb{R}^m$. This is performed by using some optimisation techniques and a generalized Ky Fan inequality. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Convexité de certaines fonctions spectrales sur les matrices hermitiennes. On montre dans cette note la convexité des fonctions $u \circ \lambda$ et plus généralement $u \circ \lambda_B$ sur l'espace des matrices hermitiennes, où B est une matrice hermitienne définie positive fixée, lorsque $u : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ est une fonction symétrique convexe, semi-continue inférieurement sur $\mathbb{R}^m$, et finie en au moins un point de $\mathbb{R}^m$, et ce en utilisant des techniques d'optimisation et une inégalité de Ky Fan généralisée. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Version française abrégée

Rappelons que pour $B \in \mathcal{M}_m(\mathbb{C})$ fixée, et pour tout $C \in \mathcal{M}_m(\mathbb{C})$, on dit que $\lambda \in \mathbb{C}$ est une valeur B-propre de C s'il existe x non nul dans $\mathbb{C}^m$ tel que $Cx = \lambda Bx$, x est alors appelé vecteur B-propre de C. On note $\lambda_B(C)$ le vecteur des valeurs B-propres de C répétées avec leur multiplicité. Pour u une fonction symétrique, on montre dans cette note dans le cas des matrices hermitiennes la convexité des fonctions $u \circ \lambda$ et plus généralement $u \circ \lambda_B$ lorsque u satisfait certaines conditions (Théorème 0.1), ce qui fournit en particulier la concavité des fonctions $F_k = \ln \sigma_k \lambda$ et plus généralement $\ln \sigma_k \lambda_B$ (Corollaire 0.3) et la convexité des domaines de définition de ces dernières fonctions (Corollaire 0.4). La preuve exposée ici est

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une alternative à celle esquissée dans [3] page 277 ; elle est basée sur la théorie des fonctions spectrales et la conjugaison de Legendre–Fenchel, et généralise la preuve d'un problème sur l'espace des matrices réelles symétriques proposée dans [4] page 300.
L'ensemble $\mathcal{H}_m(\mathbb{C})$ des matrices carrées complexes d'ordre m hermitiennes est structuré en **espace euclidien** à l'aide du produit scalaire $\ll A, B \gg = tr\left({}^t\overline{A}B\right) = tr(AB)$, dit **produit de Schur**. On note $\Gamma_0(\mathbb{R}^m)$ l'ensemble des fonctions $u : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ convexes, semi-continues inférieurement sur $\mathbb{R}^m$, et finies en au moins un point. Etant donné $u \in \Gamma_0(\mathbb{R}^m)$ symétrique et $B \in \mathcal{H}_m(\mathbb{C})$ **définie positive fixée**, on définit :

$$V_u^B : \mathcal{H}_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\}, \text{ par } C \mapsto V_u^B(C) := u(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))$$

où $\lambda_{B,1}(C) \geq \lambda_{B,2}(C) \geq ... \geq \lambda_{B,m}(C)$ désignent les **valeurs B-propres** de C chacune répétée selon sa multiplicité. On appellera de telles fonctions V_u^B, **fonctions de valeurs B-propres** ou **fonctions B-spectrales**. Notre premier but sera de déterminer la conjuguée de V_u^B en fonction de la conjuguée de u. Rappelons que la conjuguée ou transformée de Legendre–Fenchel de u est la fonction $u^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ définie par :

$$\forall s \in \mathbb{R}^m, \quad u^*(s) = \sup_{x \in \mathbb{R}^m} \left\{ \prec s, x \succ -u(x) \right\}$$

$\prec .,. \succ$ désignant le produit scalaire standard de $\mathbb{R}^m$. On prouve ici le résultat suivant :
Théorème 0.1 (Conjugaison des fonctions de valeurs B-propres) *Si $u \in \Gamma_0(\mathbb{R}^m)$ est symétrique alors :*

(i) La conjuguée u^* ($\in \Gamma_0(\mathbb{R}^m)$) est également symétrique.

(ii) Les fonctions de valeurs B-propres V_u^B et $V_{u^*}^B$ (définies comme ci-dessus) appartiennent à $\Gamma_0(\mathcal{H}_m(\mathbb{C}))$ avec $V_{u^*}^{B^{-1}} = (V_u^B)^*$, donc en particulier la fonction de valeurs B-propres V_u^B est convexe semi-continue inférieurement.

On établit pour la preuve le lemme suivant :
Lemme 0.2 (Inégalité de Ky Fan généralisée) *Soit B une matrice hermitienne définie positive fixée d'ordre m. Alors l'inégalité suivante est satisfaite :*

$$\forall C, S \in \mathcal{H}_m(\mathbb{C}), \quad tr(CS) = \sum_{i=1}^m \lambda_i(CS) \leq \sum_{i=1}^m \lambda_{B,i}(C)\, \lambda_{B^{-1},i}(S)$$

où $\lambda_{B,1}(C) \geq ... \geq \lambda_{B,m}(C)$ désigne le vecteur des valeurs B-propres de C répétées avec leur multiplicité et $\lambda_{B^{-1},1}(S) \geq ... \geq \lambda_{B^{-1},m}(S)$ le vecteur des valeurs B^{-1}-propres de S. En outre, le cas d'égalité est vérifié si et seulement s'il existe $U \in U_m(\mathbb{C})$ tel que $C = B^{\frac{1}{2}}\, {}^t\overline{U}\Big(diag(\lambda_B(C))\Big)UB^{\frac{1}{2}}$ et $S = B^{-\frac{1}{2}}\, {}^t\overline{U}\Big(diag(\lambda_{B-1}(S))\Big)UB^{-\frac{1}{2}}$ avec $\lambda_B(C) := (\lambda_{B,1}(C), ..., \lambda_{B,m}(C))$ et $\lambda_{B-1}(S) := (\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))$.
Du théorème découlent deux corollaires :
Corollaire 0.3 *Pour tout entier $k \in \{1, ..., m\}$, la fonction :*

$$F_k^B : \mathcal{H}_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\}, \ C \mapsto F_k^B(C) = \begin{cases} -\ln \sigma_k\Big(\lambda_B(C)\Big) & si\ C \in \lambda_B^{-1}(\Gamma_k) \\ +\infty & sinon \end{cases}$$

où $\lambda_B^{-1}(\Gamma_k) := \left\{C \in \mathcal{H}_m(\mathbb{C}) /\, \lambda_B(C) \in \Gamma_k\right\}$, avec $\Gamma_k := \left\{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k,\, \sigma_j(\lambda) > 0\right\}$ est convexe. Ici σ_j désigne la j-ième fonction symétrique élémentaire.
Corollaire 0.4 *Si Γ est un convexe fermé (non vide) symétrique de $\mathbb{R}^m$, alors l'ensemble $\lambda_B^{-1}(\Gamma) := \left\{C \in \mathcal{H}_m(\mathbb{C}) /\, \lambda_B(C) \in \Gamma\right\}$ est un convexe fermé de $\mathcal{H}_m(\mathbb{C})$. En particulier, $\lambda_B^{-1}\left(\overline{\Gamma_k}\right)$ est un convexe fermé de*

$\mathcal{H}_m(\mathbb{C})$ et donc l'ensemble des fonctions k-admissibles $\mathcal{A}_k := \left\{\varphi \in C^2(M,\mathbb{R})/\lambda_\omega(\omega + i\partial\bar{\partial}\varphi) \in \Gamma_k\right\}$ (au sens de [5,6] où (M,J,g,ω) est une variété kählérienne) est convexe.

1 Introduction and statement of results

Let $\mathcal{H}_m(\mathbb{C})$ be the space of complex Hermitian matrices of order m. We recall that for any two matrices B and C of $\mathcal{H}_m(\mathbb{C})$, $\lambda \in \mathbb{C}$ is called a B-eigenvalue of C if there exists $x \neq 0$ in $\mathbb{C}^m$ such that $Cx = \lambda Bx$, x is then called a B-eigenvector of C. Moreover, given a fixed positive definite matrix $B \in \mathcal{H}_m(\mathbb{C})$, the following result holds :

Proposition 1.1 *Let $C \in \mathcal{H}_m(\mathbb{C})$, then :*

(i) The spectrum of $B^{-1}C$ (i.e. the B-spectrum of C) is entirely real.

(ii) The greatest eigenvalue of $B^{-1}C$ (i.e. the greatest B-eigenvalue of C) equals $\sup_{u\neq 0} \frac{<Cu,u>}{<Bu,u>}$, where $< .,. >$ denotes the standard Hermitian product of $\mathbb{C}^m$.

(iii) $B^{-1}C$ is diagonalizable.

Since the spectrum of $B^{-1}C$ is the spectrum of the Hermitian matrix $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$, the proof is an easy adaptation of the standard one for symmetric matrices.

In this note, we will prove for u a symmetric function and in the case of Hermitian matrices the convexity of the functions $u \circ \lambda$ and more generally $u \circ \lambda_B$ when u satisfies some conditions (Theorem 1.1) and infer the concavity of the functions $F_k = \ln \sigma_k \lambda$ and more generally $\ln \sigma_k \lambda_B$ (Corollary 1.3) as well as the convexity of their sets of definition (Corollary 1.4). Our proof is an alternative to the one sketched in [3] page 277. It is based on the spectral functions theory and the Legendre–Fenchel conjugation, it generalizes the proof of a problem on the space of real symmetric matrices given in [4] page 300.

Let $\mathcal{H}_m(\mathbb{C})$ be the set of Hermitian complex square matrices of order m, $\mathcal{H}_m(\mathbb{C})$ has a structure of **Euclidian space** thanks to the following scalar product $\ll A, B \gg = tr\left({}^t\overline{A}B\right) = tr(AB)$, called **the Schur product**. Let us denote by $\Gamma_0(\mathbb{R}^m)$ the set of functions $u : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ that are convex, lower semi-continuous on $\mathbb{R}^m$, and finite in at least one point. Given $u \in \Gamma_0(\mathbb{R}^m)$ symmetric and $B \in \mathcal{H}_m(\mathbb{C})$ **positive definite**, we define :

$$V_u^B : \mathcal{H}_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\}, \text{ by } C \mapsto V_u^B(C) := u(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))$$

where $\lambda_{B,1}(C) \geq \lambda_{B,2}(C) \geq ... \geq \lambda_{B,m}(C)$ denote the **B-eigenvalues** of C repeated with their multiplicity. Such functions V_u^B are called **functions of B-eigenvalues** or **B-spectral functions**. Our first aim is to determine the conjugation for such a function V_u^B using the conjugate function of u. Let us remind that the conjugation or the Legendre–Fenchel transform of u is the function $u^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ defined by :

$$\forall s \in \mathbb{R}^m, \quad u^*(s) = \sup_{x \in \mathbb{R}^m} \left\{ \prec s, x \succ - u(x) \right\}$$

where $\prec .,. \succ$ denotes the standard scalar product on $\mathbb{R}^m$. Our main result is the following :

Theorem 1.1 (Conjugation of functions of B-eigenvalues) *Let $u \in \Gamma_0(\mathbb{R}^m)$ be symmetric, then :*

(i) The conjugate u^* ($\in \Gamma_0(\mathbb{R}^m)$) is also symmetric.

(ii) The functions of B-eigenvalues V_u^B and $V_{u^*}^B$ (defined as above) belong to $\Gamma_0(\mathcal{H}_m(\mathbb{C}))$ with $V_{u^*}^{B^{-1}} = (V_u^B)^*$, so that in particular the function of B-eigenvalues V_u^B is convex and lower semi-continuous.

Our proof requires the following lemma (named after Ky Fan like similar inequalities established in [1,2]):

Lemma 1.2 (Generalized Ky Fan Inequality) *Let B be a fixed Hermitian positive definite matrix of order m. Then the following inequality is satisfied :*

$$\forall C, S \in \mathcal{H}_m(\mathbb{C}), \quad tr(CS) = \sum_{i=1}^{m} \lambda_i(CS) \leq \sum_{i=1}^{m} \lambda_{B,i}(C)\, \lambda_{B^{-1},i}(S)$$

where $\lambda_{B,1}(C) \geq \ldots \geq \lambda_{B,m}(C)$ *denote the* B*-eigenvalues of* C *and* $\lambda_{B^{-1},1}(S) \geq \ldots \geq \lambda_{B^{-1},m}(S)$ *the* B^{-1}*-eigenvalues of* S*. Moreover, the equality case holds if and only if there exists* $U \in U_m(\mathbb{C})$ *such that* $C = B^{\frac{1}{2}}\,{}^t\overline{U}\Big(diag(\lambda_B(C))\Big)UB^{\frac{1}{2}}$ *and* $S = B^{-\frac{1}{2}}\,{}^t\overline{U}\Big(diag(\lambda_{B-1}(S))\Big)UB^{-\frac{1}{2}}$ *with* $\lambda_B(C) := (\lambda_{B,1}(C), \ldots, \lambda_{B,m}(C))$ *and* $\lambda_{B-1}(S) := (\lambda_{B^{-1},1}(S), \ldots, \lambda_{B^{-1},m}(S))$*.*

Two corollaries follow from Theorem 1.1, namely :

Corollary 1.3 *For each* $k \in \{1, \ldots, m\}$*, the function :*

$$F_k^B : \mathcal{H}_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\},\ C \mapsto F_k^B(C) = \begin{cases} -\ln \sigma_k\Big(\lambda_B(C)\Big) & \text{if } C \in \lambda_B^{-1}(\Gamma_k) \\ +\infty & \text{otherwise,} \end{cases} \tag{1}$$

$$\text{where } \lambda_B^{-1}(\Gamma_k) := \big\{C \in \mathcal{H}_m(\mathbb{C}) /\, \lambda_B(C) \in \Gamma_k\big\}, \text{with } \Gamma_k := \big\{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k,\ \sigma_j(\lambda) > 0\big\} \tag{2}$$

is convex. Here σ_j *denotes the* j*-th elementary symmetric function.*

Corollary 1.4 *If* Γ *is a (non empty) symmetric convex closed set of* $\mathbb{R}^m$*, then* $\lambda_B^{-1}(\Gamma) := \{C \in \mathcal{H}_m(\mathbb{C}) / \lambda_B(C) \in \Gamma\}$ *is a convex closed set of* $\mathcal{H}_m(\mathbb{C})$*. In particular,* $\lambda_B^{-1}\big(\overline{\Gamma_k}\big)$ *is a convex closed set of* $\mathcal{H}_m(\mathbb{C})$ *so that the set of* k*-admissible functions* $\mathcal{A}_k := \big\{\varphi \in C^2(M, \mathbb{R}) / \lambda_\omega(\omega + i\partial\bar{\partial}\varphi) \in \Gamma_k\big\}$ *(in the sense of [5,6] where* (M, J, g, ω) *is a Kähler manifold) is convex.*

In the sequel of the note, we will successively prove Lemma 1.2 (Section 2), Theorem 1.1 (Section 3) and the corollaries (Section 4).

2 Proof of the lemma

We present here a generalisation (Lemma 1.2) in terms of B-eigenvalues of Hermitian matrices of an inequality called Ky Fan inequality (see [1,2]), since it is required for the above proof of the Theorem 1.1. Here, we just indicate the proof (for details, see [5]). We first prove the theorem with $B = I$ (classical eigenvalues) by mimicking the proof of [1] (pages 17-18 and exercise 12 page 20) of a similar theorem for real symmetric matrices; the main ingredients are a Birkhoff theorem and a Hardy-Littlewood-Polya inequality. Our theorem, with general B, follows by application of the previous result to the matrices $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$ and $B^{\frac{1}{2}}SB^{\frac{1}{2}}$ with $C, S \in \mathcal{H}_m(\mathbb{C})$.

3 Proof of the theorem

Let $u \in \Gamma_0(\mathbb{R}^m)$ be symmetric. Let σ be a permutation of $\{1, \ldots, m\}$ and $s \in \mathbb{R}^m$, we have :

$$u^*(s_{\sigma(1)}, \ldots, s_{\sigma(m)}) = \sup_{x \in \mathbb{R}^m} \Big\{ \prec (s_{\sigma(1)}, \ldots, s_{\sigma(m)}), x \succ - u(x) \Big\} = \sup_{x \in \mathbb{R}^m} \Big\{ \prec s, x \succ - u(x) \Big\} = u^*(s)$$

which proves that u^* ($\in \Gamma_0(\mathbb{R}^m)$) is also symmetric. It is then possible, as done for u, to define the function of B-eigenvalues $V_{u^*}^B$. In order to show the second point of Theorem 1.1, we first prove the following equality (setting $E = \mathcal{H}_m(\mathbb{C})$ for short) :

$$\forall C \in E, \quad V_u^B(C) = \sup_{S \in E} \Big\{ tr(CS) - u^*(\lambda_{B^{-1},1}(S), \ldots, \lambda_{B^{-1},m}(S)) \Big\} \tag{3}$$

where $tr(C)$ stands for the trace of C.
Step 1 : We claim that the right-hand side of (3) is always $\geq V_u^B(C)$. Indeed, let $C \in E$ and $U \in U_m(\mathbb{C})$ be such that :

$$U(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})U^{-1} = diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))$$

so that $C = B^{\frac{1}{2}}U^{-1}diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))UB^{\frac{1}{2}}$. For each $S \in E$, we have :

$$tr(CS) = tr(B^{\frac{1}{2}}U^{-1}diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))UB^{\frac{1}{2}}S) = tr(diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1})$$

consequently :

$$\sup_{S\in E}\left\{tr(CS) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))\right\}$$
$$= \sup_{S\in E}\left\{tr(diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1}) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))\right\}.$$

Since the map : $S \in E \mapsto UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1} \in E$ is bijective, setting $\tilde{S} = UB^{\frac{1}{2}}SB^{\frac{1}{2}}U^{-1}$, we obtain :

$$\sup_{S\in E}\left\{tr(CS) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))\right\} = \sup_{\tilde{S}\in E}\Big\{tr(diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))\tilde{S})$$
$$- u^*(\lambda_{B^{-1},1}(B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}}), ..., \lambda_{B^{-1},m}(B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}}))\Big\}.$$

Now, the B^{-1}-eigenvalues of $B^{-\frac{1}{2}}U^{-1}\tilde{S}UB^{-\frac{1}{2}}$ are nothing but the eigenvalues of $U^{-1}\tilde{S}U$, hence those of $\tilde{S}$ as well, therefore :

$$\sup_{S\in E}\left\{tr(CS) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))\right\} = \sup_{\tilde{S}\in E}\left\{tr(diag(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))\tilde{S}) - u^*(\lambda_1(\tilde{S}), ..., \lambda_m(\tilde{S}))\right\}.$$

Restricting to diagonal matrices, we get :

$$\sup_{S\in E}\left\{tr(CS) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))\right\}$$
$$\geq \sup_{(s_1,...,s_m)\in\mathbb{R}^m}\left\{\sum_{i=1}^m \lambda_{B,i}(C)\, s_i - u^*(s_1, ..., s_m)\right\} = u^{**}(\lambda_{B,1}(C), ..., \lambda_{B,m}(C))$$

But $u^{**}(\lambda_{B,1}(C), ..., \lambda_{B,m}(C)) = u(\lambda_{B,1}(C), ..., \lambda_{B,m}(C)) = V_u^B(C)$ since $u \in \Gamma_0(\mathbb{R}^m)$, so step 1 is proved.

Step 2 : let us prove the reversed inequality. Given $C, S \in E$, the generalized Ky Fan inequality (Lemma 1.2) implies :

$$tr(CS) = \sum_{i=1}^m \lambda_i(CS) \leq \sum_{i=1}^m \lambda_{B,i}(C)\, \lambda_{B^{-1},i}(S)$$

hence :

$$tr(CS) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S)) \leq \sum_{i=1}^m \lambda_{B,i}(C)\, \lambda_{B^{-1},i}(S) - u^*(\lambda_{B^{-1},1}(S), ..., \lambda_{B^{-1},m}(S))$$
$$\leq \sup_{(s_1,...,s_m)\in\mathbb{R}^m}\left\{\sum_{i=1}^m \lambda_{B,i}(C)\, s_i - u^*(s_1, ..., s_m)\right\} = u^{**}(\lambda_{B,1}(C), ..., \lambda_{B,m}(C)) = V_u^B(C)$$

so the proof of the equality (3) is complete. It says exactly that :

$$V_u^B(C) = \sup_{S\in E}\left\{\ll C, S \gg - V_{u^*}^{B^{-1}}(S)\right\} = \left(V_{u^*}^{B^{-1}}\right)^*(C) \tag{4}$$

Accordingly $V_u^B = \left(V_{u^*}^{B^{-1}}\right)^*$, thus $V_u^B \in \Gamma_0(E)$ (it is convex lower semi-continuous as a conjugate, finite in at least one point because u is so). The conjugate u^* being also a symmetric function of $\Gamma_0(\mathbb{R}^m)$, the previous result applied to u^* shows that $V_{u^*}^B \in \Gamma_0(E)$ and $V_{u^*}^B = \left(V_u^{B^{-1}}\right)^*$. The proof of Theorem 1.1 is complete.

4 Proof of the corollaries

The proof of Corollary 1.3 is a direct application of Theorem 1.1 to the function :

$$u : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\},\ x = (x_1, ..., x_m) \mapsto u(x) = \begin{cases} -\ln \sigma_k(x_1, ..., x_m) & \text{if } x \in \Gamma_k \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

The proof of Corollary 1.4 goes by considering the indicatrix function $f_0 := I_\Gamma$ of the set Γ, namely :

$$f_0 := I_\Gamma : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\},\ x = (x_1, ..., x_m) \mapsto I_\Gamma(x) = \begin{cases} 0 & \text{if } x \in \Gamma \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

From the assumptions made on Γ, f_0 lies in $\Gamma_0(\mathbb{R}^m)$ and is symmetric. So Theorem 1.1 implies that the function of B-eigenvalues $V_{I_\Gamma}^B$ lies in $\Gamma_0(\mathbb{R}^m)$; in particular it is, convex lower semi-continuous. But this function is given by :

$$V_{I_\Gamma}^B : \mathcal{H}_m(\mathbb{C}) \to \mathbb{R} \cup \{+\infty\},\ C \mapsto V_{I_\Gamma}^B(C) = \begin{cases} 0 & \text{if } C \in \lambda_B^{-1}(\Gamma) \\ +\infty & \text{otherwise} \end{cases} \tag{7}$$

in other words, it coincides with $I_{\lambda_B^{-1}(\Gamma)}$, the indicatrix function of $\lambda_B^{-1}(\Gamma)$. So the latter must itself be convex lower semi-continuous. As a consequence, $\lambda_B^{-1}(\Gamma)$ is a convex closed (non empty) set of $\mathcal{H}_m(\mathbb{C})$.

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References

[1] **J.M. Borwein, A.S. Lewis**, *Convex Analysis and Nonlinear Optimization : Theory and Examples*, Springer (2000)

[2] **S. Boyd, L. Vandenberghe**, *Convex Optimization*, Cambridge University Press (2004)

[3] **L. Caffarelli, L. Nirenberg, J. Spruck**, *The Dirichlet problem for nonlinear second order elliptic equations, III : Functions of the eigenvalues of the Hessian* , Acta Math. 155 : 261-301 (1985)

[4] **J-B. Hiriart-Urruty**, *Optimisation et Analyse Convexe : Exercices corrigés*, EDP Sciences (2009)

[5] **A. Jbilou**, *Equations hessiennes complexes sur les variétés kählériennes compactes*, Thèse, Univ. Nice Sophia-Antipolis (19 février 2010), downloadable at http://tel.archives-ouvertes.fr/tel-00463111

[6] **A. Jbilou**, *Equations hessiennes complexes sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris 348 : 41-46 (2010)